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WAVE PATTERNS IN A STREAM AT NEAR-CRITICAL SPEED

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NOTATION

a'	Amplitude of solitary wave
A,B,C,D	Constants in the solitary wave equation
b'	Elevation of vortex singularity ($b = b'/\ell'$)
$f(z)$	Complex potential
F'	Drag force ($F = F'/\rho g \ell'^2$)
h'	Solitary wave depth at infinity ($h = h'/\ell'$)
k	Integration variable
ℓ'	Depth at crest(solitary wave) and upstream (singularity).
N	Outer free-surface elevation ($N = \eta'/\ell'$)
u',v'	Inner velocity components ($u = u'/(g\ell')^{\frac{1}{2}}$, $v = v'/(g\ell')^{\frac{1}{2}}$)
u'_A	Velocity at solitary wave crest ($u_A = u'_A/(g\ell')^{\frac{1}{2}}$)
U,V	Outer velocity components (dimensionless)
w	Complex velocity
x',y'	Horizontal and vertical inner coordinates ($x = x'/\ell'$, $y = y'/\ell'$)
X,Y	Outer coordinates
z	Complex variable
α	Dimensionless wave number
β	Constant in the solitary wave equation
δ'	Vortex strength ($\delta = \delta'/g^{\frac{1}{2}}\ell'^{\frac{3}{2}}$)
ϵ	Small parameter
η'	Inner free-surface elevation ($\eta = \eta'/\ell'$)

ABSTRACT

The two-dimensional free-surface flow generated by a singularity moving with near-critical speed (i.e. with Froude number referred to the water depth near unity) is solved by using the method of matched asymptotic expansions. In the vicinity of the singularity the problem is solved by an infinitesimal wave expansion (inner expansion) while at large distances from the singularity shallow water theory provides the proper solution (outer expansion). A composite expansion provides a uniformly-valid solution for the velocity components, the free-surface profile and the drag force. These two basic approaches of water-wave theory are shown to be solutions of the same flow problem, but valid in different regions. Although the inner expansion satisfies linear equations, the solution depends nonlinearly on the small parameter of the problem (the singularity strength).

INTRODUCTION

A singularity moving at constant speed in finite-depth water is considered herein. The flow is two-dimensional and it is steady when referred to a moving coordinate system. In this system the standing singularity is considered to be a perturbation of a stream of uniform velocity.

The nonlinear free-surface problem has been solved by a first order infinitesimal wave expansion for different types of singularities (Wehausen and Laitone, 1960, review the solutions).

All solutions diverge, however, when the unperturbed velocity approaches its critical value, i.e. when the Froude number based on stream depth tends to unity. It is generally assumed that the linear theory fails in this case (Stoker, 1957, p. 217).

The other basic approach, nonlinear shallow water theory, is able to predict the existence of waves of finite amplitude at near-critical speeds; but this theory cannot represent flows in the vicinity of singularities, because it is based on the assumption that the horizontal velocity component is almost uniform and much larger than the vertical one.

The purpose of this paper is to solve the problem of critical flow by using the method of matched asymptotic expansions (Van Dyke, 1964). The infinitesimal wave expansion will be shown to provide a meaningful solution in the vicinity of the singularity and will be called, consequently the inner expansion. This expansion diverges at a large distance from the singularity. There, an outer expansion, derived according to matching requirements, described the flow conditions. This outer expansion is precisely the well-known shallow water theory. A composite expansion, therefore, solves the paradoxal problem of critical flows. The results, however, go beyond this purpose; they show that the two basic theories of water waves are organically interrelated by the method of matched asymptotic expansions.

The present paper is an extension of previous work in which the method of matched asymptotic expansions was applied to free-surface flows in porous media (Dagan, 1967).

THE HOMOGENEOUS PROBLEM: THE SOLITARY WAVE

The derivation of the solitary wave by the method of matched asymptotic expansions will be discussed in the first stage.

With the variables of Figure 1 made dimensionless by referring the velocity components u, v to $(g\ell')^{\frac{1}{2}}$ and the lengths to ℓ', the exact equations of flow are

$$\left.\begin{aligned} u_x + v_y &= 0 \qquad [1] \\ u_y - v_x &= 0 \qquad [2] \end{aligned}\right\} \quad (0 < y < \eta)$$

$$\left.\begin{aligned} \tfrac{1}{2}(u^2 + v^2) + \eta &= 1 + \frac{u_A^2}{2} \qquad [3] \\ v - u\eta_x &= 0 \qquad [4] \end{aligned}\right\} \quad (y = \eta)$$

$$v = 0 \qquad (y = 0) \qquad [5]$$

$$\eta = 1 \qquad (x = 0) \qquad [6]$$

$$\eta = h \qquad (|x| = \infty) \qquad [7]$$

u_A being the velocity at the crest.

The solution for the flow in the vicinity of crest A is sought by a first-order infinitesimal wave expansion whose zero order term is a uniform flow of velocity u_o and depth $\eta_o = 1$, i.e.

$$\left.\begin{aligned} u &= u_o + \epsilon u_1 \\ v &= \epsilon v_1 \\ \eta &= 1 + \epsilon \eta_1 \\ u_A &= u_o + \epsilon u_{A_1} \end{aligned}\right\} \qquad [8]$$

ϵ being the small parameter.

By substituting the expansion [8] into Equations [1]-[6], and discarding the condition at infinity [7], one obtains the usual linearized equations

$$\left.\begin{aligned} u_{1x} + v_{1y} &= 0 \qquad [9] \\ u_{1y} - v_{1x} &= 0 \qquad [10] \end{aligned}\right\} \quad (0 < y < 1)$$

$$\left.\begin{aligned} u_o u_1 + \eta_1 &= u_o u_{A_1} \qquad [11] \\ v_1 - u_o \eta_{1x} &= 0 \qquad [12] \end{aligned}\right\} \quad (y = 1)$$

$$v_1 = 0 \qquad (y = 0) \qquad [13]$$

$$\eta_1 = 0 \qquad (x = 0) \qquad [14]$$

Considering, for the sake of simplicity, only the region $x \geq 0$, the elementary solution of Equations [9]-[ ] may be written as

$$u_1 = c_s(\cosh \alpha - \cos \alpha x \cosh \alpha y) + c_c \sin \alpha x \cosh \alpha y + u_{A_1} \qquad [15]$$

$$v_1 = -c_s \sin \alpha x \sinh \alpha y - c_c \sinh \alpha y \cos \alpha x \qquad [16]$$

$$\eta_1 = -c_s u_o \cosh \alpha\,(1-\cos \alpha x) - c_c u_o \cosh \alpha \sin \alpha x \qquad [17]$$

where α is the root of the equation

$$\frac{\tanh \alpha}{\alpha} = u_o^2 \qquad [18]$$

and $u_o \le 1$.

The solution is continued in the region $x < 0$ by replacing x by $-x$ in Equations [15]-[17]. The two arbitrary constants c_s and c_c multiply, therefore, a smooth solution at the origin and a cusped one, respectively (equation 17).

The above well-known solution is periodic and cannot satisfy the requirements of nonperiodicity and uniformity at $|x| = \infty$ expressed by Figure 1 and Equation [7]. In the vicinity of the origin, however, and for small α (i.e. u_o close to the critical speed $u_o = 1$) the solution [15]-[18] may be expanded in a power series as follows

$$u = u_o + \epsilon\left[c_c \alpha x + c_s \frac{\alpha^2(x^2-y^2+1)}{2} + u_{A_1}\right] \qquad [19]$$

$$v = \epsilon\ (-c_c \alpha y - c_s \alpha^2 xy) \qquad [20]$$

$$\eta = 1 + \epsilon \left(- c_c u_o \alpha x - c_s u_o \frac{\alpha^2 x^2}{2} \right) \qquad [21]$$

$$\frac{\tanh\ \alpha}{\alpha} = 1 - \frac{\alpha^2}{3} + \ldots = u_o^{\ 2} \qquad [22]$$

with the highest order retained being $O(\epsilon\alpha^2)$.

The solution expressed by Equations [19]-[22] becomes unbounded as $x \rightarrow \infty$, the expansion being therefore singular there. Considering this as an inner expansion, an outer expansion is sought according to the usual procedure (Van Dyke, 1964) by adopting the following outer variables

$$X = \epsilon^{\frac{1}{2}} x \qquad Y = y \qquad N = \eta \qquad U = u \qquad V = \epsilon^{\frac{1}{2}} v \qquad [23]$$

The outer variables of [23] have been selected so that for large x the quadratic term in Equation [21] becomes of the order of magnitude $O(1)$ with respect to ϵ.

The inner variables may now be expanded tentatively in an ϵ power series

$$U = U_o + \epsilon U_1 + \ldots \qquad [24]$$

$$V = \epsilon V_1 + \ldots \qquad [25]$$

$$N = N_o + \epsilon N_1 + \ldots \qquad [26]$$

The Equations [23]-[26] represent precisely the shallow water expansion of Friederichs and Keller (Friederichs, 1948).

The outer expansion of the exact Equations [1]-[6] has been carried out systematically by Laitone (Laitone, 1960) and will not be repeated here. The first order equations are the well-known shallow water equations, the solution being (Laitone, 1960)

$$U_o = N_o^{\frac{1}{2}} = \text{const.} \qquad [27]$$

$$U_1 = f(X) \qquad N_1 = -[U_o f(X) + C] \qquad V_1 = 0 \qquad [28]$$

The function f(X) satisfies the differential equation

$$f_{XX} - \frac{9}{2U_o^5} f^2 - \frac{3C}{U_o^6} f = 0 \qquad [29]$$

where C is an arbitrary constant.

Since the solution has to be used only in the outer zone, the nonlinear Equation [29] may be easily solved by iterations in the vicinity of $X = \infty$ (with $f(\infty) = 0$); but we may take advantage of the fact that the exact solution of [29] is well known being in the nonperiodical case (Laitone, 1963)

$$f(X) = -\frac{C}{U_o} \operatorname{sech}^2 \left[\left(\frac{3}{4} \frac{C}{U_o^6} \right)^{\frac{1}{2}} (X + \beta) \right] \qquad [30]$$

where β is an arbitrary constant.

The constants u_o, u_{A_1}, ϵ, c_s, c_c, C, U_o, N_o and β which appear in the inner and outer solutions have to be found by matching and by using the Equations [7] and [27]. The matching is carried out by requiring that the inner limit of the outer representation should be equal to the outer limit of the inner representation (Van Dyke, 1964).

The inner limit of the outer representation is obtained by replacing the outer variables in Equations [27]-[30] by the inner variables of Equation [23] and expanding as $\epsilon \to 0$ and X is fixed. The result for the function f is

$$f = -\frac{C}{U_o A^2}\left[1 - 2BD\epsilon^{\frac{1}{2}}x - D^2(1-3B^2)\,\epsilon x^2\right] + \ldots \qquad [31]$$

where, for, briefness, $D = \left(\frac{3}{4}\frac{C}{U_o^{\,6}}\right)^{\frac{1}{2}}$, $A = \cosh D\beta$ and $B = \tanh D\beta$.

In Equation [31] only the terms necessary for matching at the considered order have been retained.

The substitution of [31] in Equations [23]-[28], rewritten in inner variables, yields

$$N = N_o - \epsilon CB^2 - \frac{2CBD}{A^2}\,\epsilon^{3/2}x - \frac{CD^2(1-3B^2)}{A^2}\,\epsilon^2 x^2 \qquad [32]$$

$$U = U_o - \frac{\epsilon C}{U_o A^2} + \frac{2CBD}{U_o A^2}\,\epsilon^{3/2}x + \frac{CD^2(1-3B^2)}{U_o A^2}\,\epsilon^2 x^2 \qquad [33]$$

The comparison of Equations [32] and [33] with Equations [19] and [21] shows that the matching at zero order requires

$$N_o = U_o = u_o = 1 \qquad [34]$$

which means that the unperturbed flow must be near critical. Moreover, α has to be of order $\epsilon^{\frac{1}{2}}$ in order to make the matching possible at higher order. For the sake of simplicity we can take $\alpha^2 c_s = \epsilon$ and express the inner expansion [19] and [21] as

$$\eta = 1 - c_c\epsilon^{3/2}x - \frac{\epsilon^2}{2} x^2 + \ldots \qquad [35]$$

$$u = 1 + \epsilon u_{A_1} + c_c\epsilon^{3/2}x + \frac{1}{2} \epsilon^2 x^2 + \ldots \qquad [36]$$

Since terms of order $\epsilon^2 y^2$ and ϵ^2 of u appear in $\epsilon^2 U_2$ only and terms of order $\epsilon^2 y$ of v appear in $\epsilon^2 V_2$ (Laitone, 1960), they have been discarded in Equations [35] and [36].

It can be easily checked that the pair of functions

$$\left.\begin{aligned} u &= c_c x \\ v &= -c_c y \end{aligned}\right\} \qquad \left.\begin{aligned} u &= c_1 \frac{x^2 - y^2}{2} \\ v &= -c_s xy \end{aligned}\right\} \qquad [37]$$

which appear in the inner expansion [19] and [20] are exact solutions of the linearized Equations [9]-[13] when $u_o = 1$. They are the elementary first order solutions of the homogeneous linearized

equations of a free-surface flow at critical speed, counterpart of the sin and cos solutions [15] and [16] of a subcritical flow. The Equation [37] may be used directly as first order solutions of Equations [9]-[13] with $u_o = 1$. The derivation by the limiting process discussed here is useful, however, for the case of a singularity as well as for higher order expansions.

The matching of the outer terms [32] and [33] with the inner terms [35] and [36] is a matter of algebraic computation.

In the case of the solitary wave a solution which is smooth at the origin is sought. Hence, $c_c = 0$ and the corresponding outer term $2CBD/U_oA^2 = 0$, i.e. $B = 0$, $A = 1$, and $\beta = 0$.

The inner and outer expansions, written in inner variables, become in this case

$$\left.\begin{aligned} \eta &= 1 - \frac{\epsilon^2}{2} x^2 \\ u &= u_o + \epsilon u_{A_1} + \frac{\epsilon^2}{2} x^2 \\ N &= N_o - \frac{3}{4} \frac{C^2}{U_o^6} \epsilon^2 x^2 \\ U &= U_o - \frac{\epsilon C}{U_o} + \frac{3}{4} \frac{C^2}{U_o^7} \epsilon^2 x^2 \end{aligned}\right\} \quad [38]$$

The matching at first order with the supplementary conditions $U_o = N_o^{\frac{1}{2}}$ and $N_o = H$ (for $x \to \infty$) yields

$$N_o = U_o = u_o = 1 \quad \epsilon C = 1-H \quad C = (2/3)^{\frac{1}{2}} \quad \epsilon u_{A_1} = 1-H \qquad [39]$$

The complete solution of the solitary wave should be expressed by a composite expansion, but this is not necessary since the outer expansion is regular in the inner zone and coincides with the inner expansion, i.e. the overlapping zone coincides with the inner zone. Hence, the whole solution is expressed by Equations [26], [28], [30] and [39] in inner variables as

$$\eta = h + (1-h)\ \mathrm{sech}^2 \left\{ \left[\frac{3(1-h)}{4} \right]^{\frac{1}{2}} x \right\} \qquad [40]$$

which is the well known solitary wave solution. The velocities at the crest and at infinite are

$$\begin{aligned} u_A &\simeq 1 - \epsilon u_{A_1} = h \\ u_\infty &= U_o = 1 \end{aligned} \qquad [41]$$

or, in dimensional variables

$$u_A' = h' \sqrt{\frac{g}{\ell'}} = \sqrt{\frac{gh'}{1+a'/h'}}$$

$$u_\infty' = \sqrt{g\ell'} = \sqrt{gh'(1+a'/h')}$$

which is, again, a classical result.

It seems, therefore, that the method of matched asymptotic expansions does not provide any improvement in the accuracy of the shallow water solution. This feature is, however, characteristic for the solitary wave which has a uniform structure in the vicinity of the crest, but will be different in the case of a moving singularity.

The derivation of the cnoidal waves is entirely similar to that of the solitary wave and will not be considered here.

The possibility of matching the outer expansion with an inner solution which has a non-zero derivative at the origin ($c_c \neq 0$) is the key to the solution of flow past singularities.

FLOW PAST A SINGULARITY

A submerged vortex in a uniform, unperturbed stream will be considered for the sake of definiteness. Any other type of singularity may be treated in a similar way (Figure 2).

It is worthwhile for the following derivations to discuss first the radiation condition. In both subcritical and supercritical flows the flow is unperturbed at $x = -\infty$ (Stoker, 1957). Adopting the same condition in the case of a near-critical flow, it is convenient to take, $\eta_o = 1$ and u_o in the inner expansion as the unperturbed depth and velocity at $x = -\infty$ rather than at $x = 0$. Since the infinitesimal wave expansion will be shown to be regular, under the above radiation condition, in the region $x < 0$, the expansion may still be called the "inner" expansion. Hence, the first order linearized expansion, similar to Equation [8] is

$$\left.\begin{aligned} u &= u_o + \delta(\epsilon)u_1 \\ v &= \delta(\epsilon)v_1 \\ \eta &= 1 + \delta(\epsilon)\eta_1 \end{aligned}\right\} \qquad [42]$$

The gauge function $\delta(\epsilon)$ represents the vortex strength ($\delta(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$). Its form will be found from the requirements of matching with the already found outer solution [32] and [33]. One could obviously start the inner expansion with an ϵ power series but an unknown gauge function has to be used in this case in the outer expansion in order to make the matching possible.

The variables of the inner expansion [42] satisfy the Equations [9], [10], [12] and [13], Equations [11] and [14 being replaced by

$$u_o u_1 + \eta_1 = 0 \qquad (y = 1) \qquad [43]$$

$$\eta_1 = 0 \qquad (x = -\infty) \qquad [44]$$

As usual the velocity components are separated in regular and singular parts

$$u_1 = u_1^{\,r} + u_1^{\,s} \qquad v_1 = v_1^{\,r} + v_1^{\,s} \qquad [45]$$

The singular term is selected to represent the potential flow due to a vortex located at $x = 0$, $y = b$ and confined between two solid boundaries $y = 0$ and $y = 1$ (Figure 2).

The complex potential of this flow is

$$f_1^s\ (z) = \frac{1}{2\pi i}\ \ell n \frac{\sinh\ \frac{\pi}{2}(z+ib)}{\sinh\ \frac{\pi}{2}(z-ib)} \qquad [46]$$

where $z = x + iy$ and $w_1{}^s = u_1{}^s - iv_1{}^s = \frac{df_1{}^s}{dz}$.

The velocity along the upper boundary is

$$u_1^s(x,1) = \frac{1}{2}\ \frac{\sin\ \pi}{\cosh\ \pi x + \cos\ \pi} \qquad [47]$$

The regular parts of the velocity components have to satisfy the equation

$$u_o{}^2 u_{1x}^r + v_1^r = - u_o{}^2 u_{1x}^s \qquad (y = 1) \qquad [48]$$

which is obtained from Equations [43] and [12] by the elimination of η_1. The solution may be represented by integral Fourier transforms (Wehausen and Laitone, 1960) as

$$w_1^r(z) = u_1^r - iv_1^r$$

$$= -\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\sin \pi b}{\cosh \pi s + \cos \pi b}\cos k(z-s)\ PV\int_{-\infty}^{\infty}\frac{k}{k \cosh k - 1/u_o^2 \sinh k}\,dk$$

[49]

where $u_o^2 \leq 1$. Equation [49] will be integrated for $u_o < 1$ and u_o will tend to its critical value $u_o = 1$ in the final result, after imposing the radiation condition.

The details of the integration for u_1^r (x,1) are given in the Appendix. The final result for $u_1(x,1)$ is

$$u_1(x,1) = u_1^r(x,1) + u_1^s(x,1)$$

$$= \frac{k_o \sinh k_o b \sin k_o x}{\sinh k_o \left(1 - \frac{1}{u_o^2} + u_o^2 k_o^2\right)} - \sum_{m=1}^{\infty}\frac{k_m \sin k_m b\ e^{-k_m x}}{\left(\frac{1}{u_o^2} + k_m^2 u_o^2 - 1\right)\sin k_m} + c \quad [50]$$

for $x \geq 0$. For $x \leq 0$, x has to be replaced by -x in Equation [50]. c is an arbitrary constant. In order to ensure the radiation requirement, the first term is removed from Equation [50] for $x < 0$ by adding to u_1 the proper solution of the homogeneous equation [48] (Lamb, 1945), which is, in the present case

$$\frac{k_o \sinh k_o b \sin k_o x}{\sinh k_o \left(1 - \frac{1}{u_o^2} + u_o^2 k_o^2\right)}.$$

The limiting case of $u_o \rightarrow 1$ may be computed by taking into account that k_o is a root of the following equation

$$\frac{\tanh k}{k} = \frac{k - k^3/3 + \ldots}{k} = u_o^2 \qquad [51]$$

Hence $u_o^2 = 1 - k_o^2/3 + \ldots$ and the final expressions of $u_1\,(x,1)$ when $u_o \rightarrow 1$ become

$$u_1(x,1) = 3bx - \sum_{m=1}^{\infty} \frac{\sin k_m b \; e^{-k_m x}}{k_m \sin k_m} \qquad (x \geq 0) \qquad [52]$$

$$u_1(x,1) = - \sum_{m=1}^{\infty} \frac{\sin k_m b \; e^{k_m x}}{k_m \sin k_m} \qquad (x \leq 0) \qquad [53]$$

the higher terms (in k_o^2, k_o^4 ...) being neglected.

The complete solution of η, according to Equations [42], [43], [44], [52] and [53] is

$$\eta = 1 - 3b\delta x + \sum_{m=1}^{\infty} \frac{\sin \kappa_m b \; e^{-k_m x}}{k_m \sin k_m} \qquad (x \geq 0) \qquad [54]$$

$$\eta = 1 + \sum_{m=1}^{\infty} \frac{\sin k_m b \; e^{k_m x}}{k_m \sin k_m} \qquad (x \leq 0) \qquad [55]$$

where $\tan k_m = k_m$ and $k_m > 0$.

The inner solution is singular as $x = \infty$. Using the previous outer expansion (Equation 23), the inner solution may be matched with the outer solution [32] and [33] in the region $x > 0$.

The outer limit of the inner representation is found from Equation [55], through the usual procedure, to be

$$\eta = 1 - 3b\delta(\epsilon)x \qquad [56]$$

where terms containing $e^{-k_m \epsilon^{-\frac{1}{2}} x}$ have been discarded since they are exponentially small.

The outer limit of the inner representation of $u(x,y)$ should be determined by integrating Equation [46] for arbitrary y. It is known, however, that at $x \gtrsim 1$ the u distribution is almost uniform and $u(x,1)$ may be substituted for $u(x,y)$. Hence, according to Equations [42] and [54]

$$u = u_o + 3b\delta x - \delta \sum_{m=1}^{\infty} \frac{\sin k_m b}{k_m \sin k_m} e^{-k_m x} \qquad [57]$$

and the outer limit is

$$u = u_o + 3b\delta x. \qquad [58]$$

The matching of the inner solution [56] and [58] with the outer solution [32] and [33] at zero order implies $N_o = U_o = 1$. The matching at ϵ order requires that U_o and N_o should be ϵ dependent (a similar situation occurs in the problem of inviscid flow past an airfoil (Van Dyke, 1964)). Hence,

$$N_o = 1 + \epsilon C B^2$$
$$U_o = u_o + \frac{\epsilon C}{U_o A^2} \qquad [58a]$$

These two relationships and Equation [27] show that, at the order considered here,

$$u_o = 1 + \epsilon C \left(\frac{3}{2} B^2 - 1 \right) \qquad [59]$$

where the relationship $1/A^2 = 1 - B^2$ has been used.

The matching of the terms dependent on x provides the additional relationship

$$C\epsilon = 3^{\frac{1}{3}} \left(\frac{A^2}{B} b\delta \right)^{\frac{2}{3}}. \qquad [60]$$

From Equations [59] and [60] we finally find

$$\frac{\left(1 - \frac{3B^2}{2}\right)^3}{B^2 \, (1-B^2)^2} = \frac{(1-u_o)^3}{3(b\delta)^2} \qquad [61]$$

$$\epsilon C = \frac{3^{1/3}}{\left[B(1-B^2)\right]^{2/3}} \quad (b\delta)^{2/3} \tag{62}$$

Equations [61] and [62] permit the determination of B (also A and β) and ϵC as functions of δ and $1-u_o$.

Two extreme cases are of particular interest: In a first case assume that $\delta \rightarrow 0$, but $1-u_o \neq 0$. Three solutions are then possible: (i) $B = 1$, i.e. $\beta = \infty$, which means a uniform flow; (ii) $B = 0$ and $\delta/B = 0$ which is again a uniform flow since $\epsilon C = 0$; (iii) $B = 0$ and $\delta/B \neq 0$, which represents a solitary wave. The last case is nontrivial but has to be excluded from physical considerations. It has been considered as a mathematical possibility by Filippov (1960) using other methods.

In the second case $u_o = 1$ and $\delta \neq 0$, which means that the unperturbed upstream flow is exactly critical. In this case

$$B^2 = \frac{2}{3} \qquad A^2 = 3 \qquad \epsilon C = 3\left(\frac{3}{2}\right)^{1/3} (b\delta)^{2/3} \tag{63}$$

The outer solution for the free-surface profile in the general case according to Equations [26], [28] and [30], is given by

$$\eta = N_o - \epsilon C \tanh^2 \left[\left(\frac{3C}{4U_o^{\,6}}\right)^{\frac{1}{2}} (\epsilon^{\frac{1}{2}} x + \beta)\right] \tag{64}$$

All the parameters appearing in Equation [64] may be determined as functions of $(1-u_o)$ and δ from Equations [58], [59], [61] and [62]. A composite expansion with a uniform representation of η may be written by combining the inner and outer solutions (Van Dyke, 1964). In the simple case $u_o = 1$ the composite expansion of η, for $x \geq 0$, is

$$\eta = 1 + 2\left(\frac{3}{2}\right)^{\frac{1}{3}}(b\delta)^{\frac{2}{3}} + \delta\sum_{m=1}^{\infty}\frac{\sin k_m b}{k_m \sin k_m} e^{-k_m x} -$$

$$-3\left(\frac{3}{2}\right)^{\frac{1}{3}}(b\delta)^{\frac{2}{3}} \tanh^2\left[\left(\frac{3b\delta}{2}\right)^{\frac{1}{3}} x + \cosh^{-1} 3^{\frac{1}{2}}\right] \qquad [65]$$

In Figure 3 the shape of the free-surface is represented for the particular case $b = 0.6$ and $\delta = 0.01$. The series in Equation [65] has been summed by using an electronic computer with the values of k_m from Carslaw and Jaeger (1959). The matching is possible, at the order considered here, only for $\delta << 1$.

The drag force on the singularity may be found by a momentum balance between the sections $x = -\infty$ and $x = \infty$. The flow depth and velocity at $x = +\infty$ (downstream) are

$$\left.\begin{aligned} \eta &= N_o - \epsilon C = 1 - \frac{\epsilon c}{A^2} \\ u &= U_o = 1 + \frac{\epsilon c}{2B^2} \end{aligned}\right\} \qquad [66]$$

The dimensionless drag forces is, therefore, given at ϵ order, by

$$F = \frac{1}{2} - u_o^2 - \frac{1}{2}\left(1 - \frac{\epsilon C}{A^2}\right)^2 + \left(u_o + \frac{\epsilon C}{A^2}\right)^2 \left(1 - \frac{\epsilon C}{A^2}\right) = \frac{2\epsilon C}{A^2} \qquad [67]$$

At the critical speed $u_o = 1$, F has the simple expression

$$F = 2\left(\frac{3}{2}\right)^{1/3} (b\delta)^{2/3} \qquad [68]$$

or in a dimensional form

$$\frac{F'}{\rho g \ell'^2} = 2\left(\frac{3}{2}\right)^{1/3} \left(\frac{b'\delta'}{g^{\frac{1}{2}} \ell'^{3/2}}\right)^{2/3} \qquad [69]$$

DISCUSSION OF RESULTS AND CONCLUSIONS

Three basic lengths are present in the problems discussed in the foregoing sections: the flow depth ℓ', the amplitude (associated with ϵ) and the ratio $u_o'^2/g$. In the infinitesimal wave expansion the last two lengths are referred to ℓ' and the periodic solution of the homogeneous equations in the subcritical range provides the well-known dispersion relationship between $Fr = u_o'/(g\ell')^{\frac{1}{2}}$ and the wave length L' (Equation 18). For small values of α, where $\alpha = 2\pi\ell'/L'$, Equation [18] yields

$$\left(\frac{2\pi\ell'}{L'}\right)^2 \cong 3(1-Fr^2).$$

As $Fr \rightarrow 1$, $L' \rightarrow \infty$ and the homogeneous problem has a non-periodical solution divergent at infinity (Equations 37). The shallow water solution is found by relating x' to L' rather than to ℓ' and by imposing a nonlinear relationship between the amplitude and the wavelength (in the case of the solitary wave $\alpha \sim \epsilon^{\frac{1}{2}}$). In this sense the results obtained by using the method of matched asymptotic expansions are related to other derivations of the first order equations of shallow flow (see, for instance, Benjamin (1967), p. 561-562).

Although the flow has been found in the vicinity of the singularity by linear equations, the dependence of the solution on the problem's small parameter (1-h in the case of the solitary wave and δ is that of the singularity) is nonlinear, even at first order (Equations [40], [65] and [68]). In this sense the solutions may be regarded as nonlinear.

The solutions for the higher order terms of the expansions and their matching should provide very valuable information and insight into the problem, but the computations become tedious as the order is increased in the inner expansions.

The method of matched asymptotic expansions proves itself to be a powerful tool in solving nonlinear problems of fluid mechanics in general, and of water waves in particular.

APPENDIX

THE INTEGRATION OF EQUATION [49]

One of the integrals of [49] may be evaluated in the following closed form (Gradshteyn and Ryzhik, pp. 505)

$$\int_{-\infty}^{\infty} \frac{\cos k\,(z-s)}{\cosh \pi s + \cos \pi b}\, ds = 2\,\frac{\sinh kb \cos kz}{\sin \pi b \sinh k} \qquad [70]$$

Hence, Equation [49] becomes

$$w_1^{\,r}(z) = u_1^{\,2} - i v_1^{\,r} = -\frac{1}{2\pi}\, PV \int_{-\infty}^{\infty} \frac{k \sinh kb \cos kz}{\sinh k\left(k \cosh k - \frac{1}{u_o^{\,2}} \sinh k\right)}\, dk$$

[71]

On the upper boundary z = x + i the real part of [72] is

$$u_1^{\,r}(x,1) = -\frac{1}{2\pi}\, PV \int_{-\infty}^{\infty} \frac{k \sinh kb\, e^{ikx}}{\sinh k\left(k - \frac{1}{u_o^{\,2}} \tanh k\right)}\, dk \qquad [73]$$

The integration of Equation [73] follows Lamb's procedure. For x > 0 the integration is carried out in the upper k complex plane. The poles of the integrand have the following locations:

two poles $k = \pm k_o$ on the real axes, roots of the equation $\tanh k = u_o^2 k$; a row of poles on the imaginary axes $k = ikm$ $(m = 1,2,\ldots)$, roots of the equation $\tan k = u_o^2 k$ and a row of poles $k_n = in\pi$ $(n = 1,2,\ldots)$.

The contribution of the last series of poles is

$$\sum_{n=1}^{\infty} \frac{\sin k_n b \; e^{-k_n x}}{\cos k_n}$$

The same sum is obtained from the integral

$$- \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sinh kb \; e^{ikx}}{\sinh k} \, dk$$

which may be integrated in a closed form (Gradshteyn and Ryzhik, p. 507), the result being

$$\sum_{n=1}^{\infty} \frac{\sin k_n b \; e^{-k_n x}}{\cos k_n} = - \frac{1}{2} \frac{\sin \pi b}{\cosh \pi x + \cos \pi b}$$

The contribution of the poles k_m is given by the sum of residues, i.e.

$$- \sum_{m=1}^{\infty} \frac{k_m \sin k_m b \, e^{-k_m x}}{\left(\frac{1}{u_o^2} - 1 + k_m^2 u_o^2\right) \sin k_m}$$

The contribution of the poles $\pm k_o$ is evaluated by integrating [73] along two semi-circles, in order to obtain the principal value. The result is

$$\frac{k_o \sinh k_o b \sin k_o x}{\sinh k_o \left(1 - \frac{1}{u_o^2} + u_o^2 k_o^2\right)}$$

These partial results are summed in Equation [50]

When $x < 0$ the results are similar, and are obtained from those above by replacing x by $-x$.

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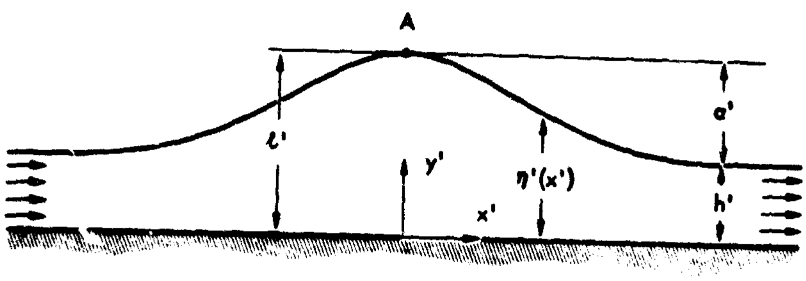

FIGURE 1 - THE SOLITARY WAVE

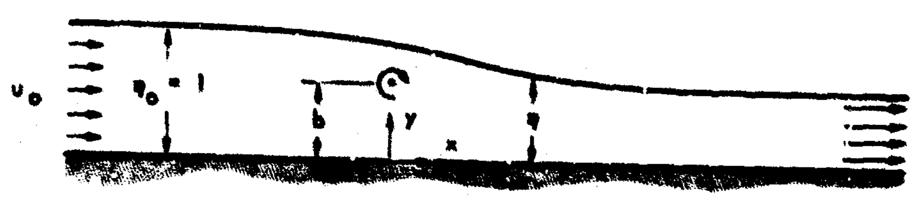

FIGURE 2 - A VORTEX IN A UNIFORM STREAM

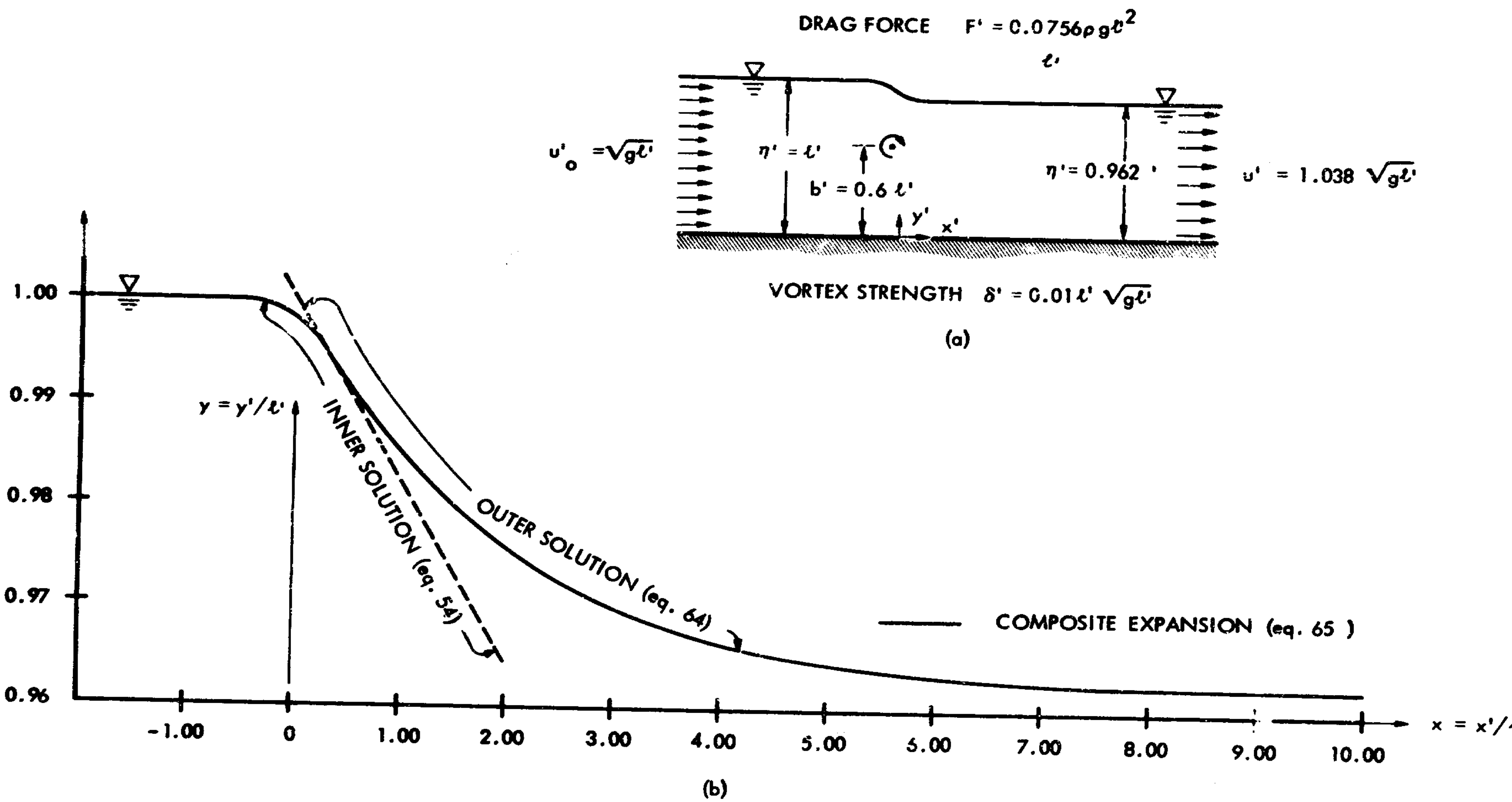

FIGURE 3 - A VORTEX IN A CRITICAL SPEED STREAM
(a) SCHEMATICAL REPRESENTATION OF THE FREE SURFACE PROFILE
(b) DETAILED REPRESENTATION IN THE MATCHING REGION

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The two-dimensional free-surface flow generated by a singularity moving with near-critical speed (i.e. with Froude number referred to the water depth near unity) is solved by using the method of matched asymptotic expansions. In the vicinity of the singularity the problem is solved by an infinitesimal wave expansion (inner expansion) while at large distances from the singularity shallow water theory provides the proper solution (outer expansion). A composite expansion provides a uniformly-valid solution for the velocity components, the free-surface profile and the drag force. These two basic approaches of water-wave theory are shown to be solutions of the same flow problem, but valid in different regions. Although the inner expansion satisfies linear equations, the solution depends nonlinearly on the small parameter of the problem (the singularity strength).

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